No mapping with n fixed n-points.

Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $n \geq 2$ be positive integer and let $g : D \to D$ be mapping, such that n times iterated mapping g_n have exactly n fixed points. Prove that there is no mapping $f : D \to D$ such that $f_n = g$. (For any $h : D \to D$ and $h_0 := Id_D$ and $h_n = h \circ h_{n-1}, n \in \mathbb{N}$).

Solution.

Definition.

We say that $x \in D$ is fixed point of order n of mapping $h : D \to D$ if $h_n(x) = x$ and $h_k(x) \neq x, k = 1, \ldots, n-1$ (or shortly x is fixed n –point of h).

Fixed point x of h we also will call fixed point of order one,or fixed 1-point,

For order of fixed point x of h we also will use notation $ord_h(x)$.

Thus x is fixed n –point, or $ord_h(x) = n$ iff x is fixed point of h_n and not fixed point of h_k for all $k < n$. Obvious that if x is fixed m –point of h then x is also is fixed point of h_n for any n divisible by m.

For any function h defined on D we denote via $\mathcal{F}_n(h)$ the set of all fixed n –points of h on D.

Accordingly to condition of problem consider now mapping $g : D \to D$ which have exactly n fixed n –points, that is $\mathcal{F}_n := \mathcal{F}_n(g)$ contain exactly n elements.

Since for any $x \in \mathcal{F}_n$ we have $g_n(g(x)) = g(g_n(x)) = g(x)$ and $g_k(g(x)) \neq g(x), k < n$ (because otherwise, if for some $1 \leq k < n$ holds $g_k(g(x)) = g(x) \Leftrightarrow g_{k+1}(x) = g(x)$ then $g_{n-k-1}(g_{k+1}(x)) = g_{n-k-1}(g(x)) \Leftrightarrow x = g_{n-k}(x)$, that is the contradiction).

Hence, $g(x) \in \mathcal{F}_n$ for any $x \in \mathcal{F}_n$ and, therefore, $g(\mathcal{F}_n) \subset \mathcal{F}_n$.

Moreover, we will prove that restriction g on $\mathcal{F}_n$ is injection and, moreover, bijection since $g(\mathcal{F}_n) \subset \mathcal{F}_n$. Indeed, if $x_1, x_2 \in \mathcal{F}_n$ and $g(x_1) = g(x_2)$ then $g_{n-1}(g(x_1)) = g_{n-1}(g(x_2)) \Leftrightarrow$ $g_n(x_1) = g_n(x_2) \Leftrightarrow x_1 = x_2$.

Note that for any $x \in \mathcal{F}_n$ and any k, l such that $1 \leq k < l \leq n-1$ holds $g_k(x) \neq g_l(x)$. Indeed, if $g_k(x) = g_l(x)$ then $g_{n-l}(g_k(x)) = g_{n-l}(g_l(x)) \Leftrightarrow g_{n-l+k}(x) = g_n(x) \Leftrightarrow$ $g_{n-l+k}(x) = x$ and it is contradiction, because $1 \leq n-l+k < n$.

Since $0 \leq k \leq n-1$ and $\{x, g_1(x), \ldots, g_{n-1}(x)\} \subset \mathcal{F}_n$ then $\mathcal{F}_n = \{x, g_1(x), \ldots, g_{n-1}(x)\}$ for any $x \in \mathcal{F}_n$.

Suppose that there is mapping $f : D \to D$ such that $f_n = g$. Then $f \circ g = f \circ f_n = f_n \circ f = g \circ f$. Also note that for any $x \in \mathcal{F}_n$ holds $f(x) \in \mathcal{F}_n$. Indeed, since $x \in \mathcal{F}_n$ then $g_n(f(x)) = f(g_n(x)) = f(x)$. Moreover, for any $x \in \mathcal{F}_n$ and any $k < n$ holds $g_k(f(x)) \neq f(x)$ because, otherwise, if there are $x \in \mathcal{F}_n$ and $k < n$ such that $g_k(f(x)) = f(x)$ then $f(g_k(x)) = f(x) \Leftrightarrow f_{n-1}(f(g_k(x))) = f_{n-1}(f(x)) \Leftrightarrow$ $f_n(g_k(x)) = f_n(x) \Leftrightarrow g(g_k(x)) = g(x)$. Since $g_k(x), x \in \mathcal{F}_n$ and g is injection on $\mathcal{F}_n$ we obtain $g_k(x) = x$, that is the contradiction. Hence, $f(x)$ is fixing n –point of g, for any $x \in \mathcal{F}_n$ and, therefore, $f(\mathcal{F}_n) \subset \mathcal{F}_n$.

So now we can consider f and g only on $\mathcal{F}_n$. Let $a \in \mathcal{F}_n$ then $\mathcal{F}_n = \{a, g_1(a), \ldots, g_{n-1}(a)\}$

Note that $f(a) \neq a$ (supposition $f(a) = a$ yield $f_n(a) = a \Leftrightarrow g(a) = a$, that

is the contradiction).Then $f(a) = g_k(a)$ for some $k = 1, 2, \ldots, n-1$.
Noting that $f_2(a) = f(g_k(a)) = g_k(f(a)) = g_k(g_k(a)) = g_{2k}(a)$ we will prove by Math. Induction that $f_m(a) = g_{mk}(a)$ for any positive integer m.
In supposition $f_m(a) = g_{mk}(a)$ we obtain $f(f_m(a)) = f(g_{mk}(a)) \iff$
$f_{m+1}(a) = g_{mk}(f(a)) \iff f_{m+1}(a) = g_{mk}(g_k(a)) \iff f_{m+1}(a) = g_{(m+1)k}(a)$.
In particular, since $g_n(a) = a$ we get $f_n(a) = g_{nk}(a) \iff g(a) = a$, that is contradiction.